3.1 Magnetic Fields

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**Oersted and Ampere**

The idea that the Earth had some property that made lodestone (magnetic) needles align in a N-S direction is the root of the concept of a magnetic force field. Oersted found that a wire carrying current also affected a compass needle and so established that it is charges in motion that create the magnetic force field. Actually the fact is that charges in motion produce a force on other charges in motion and we describe this phenomenon by introducing the concept of magnetic field.

Ampere formulated the idea of the force field when he found that there was a force created between two current carrying wires:
Force / unit length = \( k \frac{I_a I_b}{r} \)

In SI units, \( k = \frac{\mu_0}{2\pi} \) where \( \mu_0 = 4\pi \times 10^{-7} \)

Later this was generalized so the force could be calculated between two current carrying circuits \( a \) and \( b \).

The definition of magnetic induction, \( B \)

If we think of this force as something happening to circuit \( b \) caused by the total effect of circuit \( a \), we could rewrite the expression for \( F_{ab} \) as

\[
F_{ab} = \frac{\mu_0}{4\pi} I_a I_b \oint_{ab} \frac{d\vec{l}_b \times (d\vec{l}_a \times \vec{r}_l)}{r^2}
\]

\[
F_{ab} = I_b \oint_{b} d\vec{l}_b \times \left[ \frac{\mu_0}{4\pi} I_a \oint_{a} \frac{(d\vec{l}_a \times \vec{r}_l)}{r^2} \right]
\]
The term in square brackets is now a property of \( \mathbf{a} \); it is something created by circuit \( \mathbf{a} \) which interacts with the current in circuit \( \mathbf{b} \) to produce the force on \( \mathbf{b} \). It is defined as a 'field' and called the magnetic induction \( B \).

In S.I units, \( B \) is Tesla (T). In c.g.s units, \( B \) is in Gauss \((1.0 \text{T} = 10^4 \text{ gauss})\). [In many magnetic surveys a c.g.s unit called the gamma (\( \gamma \)) is used. \( 1.0 \gamma = 10^{-5} \text{ gauss} = 10^{-9} \text{T} \) \((1.0 \text{nT})\)].

The force on an element of current then has the form:

\[
d\mathbf{F}_b = I_b d\mathbf{l}_b \times \mathbf{B}
\]

and if there is a current density, \( J \), i.e. a volume distribution of current, the expression for \( B \) becomes:

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \mathbf{r}}{r^2} dV
\]

This simple formula for \( B \) was derived by Biot and Savart. The direction of the field follows the right hand rule - thumb of right hand in direction of current then fingers point in direction of \( B \).

From these definitions and a bit of potential theory, we can find the field from a small loop of wire of radius \( a \) carrying current \( I \) from:
\[ \vec{B} = \frac{\mu_0}{4\pi} \nabla \left( \vec{m} \cdot \nabla \left( \frac{1}{r} \right) \right) \quad (3.1.2) \]

or, expanded

\[ \vec{B} = \frac{\mu_0}{4\pi} \frac{2(\pi a^2 I)}{r^3} \cos \theta \hat{r} + \frac{\mu_0}{4\pi} \frac{\pi a^2 I}{r^3} \sin \theta \hat{\theta} \quad (3.1.3) \]

where

\[ \vec{m} = I \pi a^2 = IA \]

This small loop of current is called a magnetic dipole and \( \vec{m} \) is called the dipole moment of the loop. \( r \) and \( \theta \) are unit vectors in the radial and \( \theta \) directions respectively. If you plot the field lines from such a dipole source you get:

\[ B_r, \text{ at } \theta = 0^\circ, = \frac{\mu_0}{4\pi} \frac{2m}{r^3} \]

\[ B_\theta, \text{ at } \theta = 90^\circ, = \frac{\mu_0}{4\pi} \frac{m}{r^3} \]
Magnetic fields in matter

In a simplistic classical view, each atom of matter, surrounded by its shell of electrons, looks like a small loop of current and so has a magnetic moment. If the atoms are randomly oriented, the net moment is zero. If the atoms line up, there will be a net moment, $M$, due to a finite loop of magnetization current $I_m$ as shown in the sketch:

\[ M = I_m A \]

Rigorously, the moment is computed for currents flowing around a volume $V$ and the magnetization $M$ is the moment per unit volume. The individual loops may be lined up by an external field (like a compass needle) - this is paramagnetism. They may be lined up opposite to the inducing field - diamagnetism. They may spontaneously line up in small groups called domains - ferromagnetism. The domains in turn may be lined up by an external field. In typical minerals the paramagnetic effect is dominant but weak. Minerals containing iron, particularly magnetite $\text{Fe}_3\text{O}_4$, are strongly ferromagnetic and account for the
magnetization encountered in rocks. When there is magnetization, there is an additional current \( I_m \), or current density \( J_m \), which is added to any conventional current that might be present. In ferromagnetic materials the magnetization \( M \) is created by the application of a magnetic field -- an applied field causes the domain magnetizations to line up. This adds a magnetization current \( J_m \), to any other current, \( J \), that might have been there so \( B \) is now a function of \( J \) and \( J_m \).

\[
\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\left( \vec{J} + \vec{J}_m \right)}{r^2} \times \hat{\eta} \, dV
\]

With a bit of vector calculus we can show from (1) that

\[
\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{(3.1.4)}
\]

and with magnetization

\[
\vec{\nabla} \times \vec{B} = \mu_0 \left( \vec{J} + \vec{J}_m \right)
\]

and with a similar approach to that used in defining \( J_m \) and \( M \) we find

\[
\vec{\nabla} \times \vec{M} = \vec{J}_m \quad \text{(3.1.5)}
\]

and combining with (4) we get

\[
\vec{\nabla} \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}
\]

The quantity, \( B/\mu_0 - M \), whose curl is the 'free' current density is defined as \( H \) - the magnetic field intensity.

\[
\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}
\]

and so \( B = \mu_0( H + M ) \) \quad ( H and M have the same units)

For historical reasons the statement that the induced magnetization depends on the field has been written.
\[ M = \chi_m H \]

where \( \chi_m \) is the magnetic susceptibility. Because \( M \) and \( H \) are in the same units (Amp/m) \( \chi_m \) is dimensionless but, be careful,

\[ \chi_m \text{ SI units} = 4\pi \chi_m \text{ cgs units} \]

There is one more definition to complete this summary description of magnetic field.

\[
B = \mu_0 ( H + M ) \\
= \mu_0 ( H + \chi_m H ) \\
= \mu_0 ( 1 + \chi_m )H
\]

so \( B = \mu H \) and \( \mu \) is the magnetic permeability.

The relationship between \( M \) and \( H \) is complex and nonlinear and is described by a hysteresis curve, viz:

![Hysteresis Curve](image.png)